

WEIGHTED SEMIGROUP MEASURE ALGEBRA AS A WAP-ALGEBRA

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A Banach algebra \mathfrak{A} for which the natural embedding $x \mapsto \hat{x}$ of \mathfrak{A} into $WAP(\mathfrak{A})^$ is bounded below; that is, for some $m \in \mathbb{R}$ with $m > 0$ we have $\|\hat{x}\| \geq m\|x\|$, is called a WAP-algebra. Through we mainly concern with weighted measure algebra $M_b(S, \omega)$, where ω is a weight on a semi-topological semigroup S . We study those conditions under which $M_b(S, \omega)$ is a WAP-algebra (respectively dual Banach algebra). In particular, $M_b(S)$ is a WAP-algebra (respectively dual Banach algebra) if and only if $wap(S)$ separates the points of S (respectively S is compactly cancellative semigroup). We apply our results for improving some older results in the case where S is discrete.*

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1. Introduction and Preliminaries

Throughout this paper, we study those conditions under which $M_b(S, \omega)$ is either a WAP-algebra or a dual Banach algebra. Our main result in section 2 is that for a locally compact topological semigroup and a continuous weight ω on S , the measure algebra $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$ if and only if for all compact subsets F and K of S , the maps $\frac{\chi_{F^{-1}K}}{\omega}$ and $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity. This improved the result of Abolghasemi, Rejali, and Ebrahimi Vishki [1] to include the case where S is not necessarily discrete. As a consequence in non-weighted case, we conclude for a locally compact topological semigroup S , the measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if S is a compactly cancellative semigroup. The

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later result improved the well known result of Dales, Lau and Strauss [7, Theorem 4.6], $\ell_1(S)$ is dual Banach algebra with respect to $c_0(S)$ if and only if S is weakly cancellative semigroup.

Section 3 is devoted to study *WAP*-algebras on a semigroup S . For every weighted locally compact semi-topological semigroup (S, ω) , $M_b(S, \omega)$ is a WAP-algebra if and only if the evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(wap(S, 1/\omega))$. Our main result of this section is that $M_b(S)$ is WAP-algebra if and only if $wap(S)$ separate the points of S . If $C_0(S, 1/\omega) \subseteq wap(S, 1/\omega)$ then $wap(S, 1/\omega)$ separate the points of S . Thus $M_b(S, \omega)$ is a WAP-algebra. We may ask whether, if $M_b(S, \omega)$ is a WAP-algebra then $C_0(S, 1/\omega) \subseteq wap(S, 1/\omega)$. We answer to this question negatively by a counter example. Then we exhibit some necessary and sufficient condition for $c_0(S) \subseteq wap(S)$. we end the paper by some examples which show that our results cannot be improved.

The dual \mathfrak{A}^* of a Banach algebra \mathfrak{A} can be turned into a Banach \mathfrak{A} -module in a natural way, by setting

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle \quad (a, b \in \mathfrak{A}, f \in \mathfrak{A}^*).$$

A *dual Banach algebra* is a Banach algebra \mathfrak{A} such that $\mathfrak{A} = (\mathfrak{A}_*)^*$, as a Banach space, for some Banach space \mathfrak{A}_* , and such that \mathfrak{A}_* is a closed \mathfrak{A} -submodule of \mathfrak{A}^* ; or equivalently, the multiplication on \mathfrak{A} is separately weak*-continuous. We call \mathfrak{A}_* the predual of \mathfrak{A} . It should be remarked that the predual of a dual Banach algebra need not be unique, in general (see [5, 10]); so we usually point to the involved predual of a dual Banach algebra.

A functional $f \in \mathfrak{A}^*$ is said to be *weakly almost periodic* if $\{f \cdot a : \|a\| \leq 1\}$ is relatively weakly compact in \mathfrak{A}^* . We denote by $WAP(\mathfrak{A})$ the set of all weakly almost periodic elements of \mathfrak{A}^* . It is easy to verify that, $WAP(\mathfrak{A})$ is a (norm) closed subspace of \mathfrak{A}^* .

It is known that the multiplication of a Banach algebra \mathfrak{A} has two natural but, in general, different extensions (called Arens products) to the second dual \mathfrak{A}^{**} each turning \mathfrak{A}^{**} into a Banach algebra. When these extensions are equal, \mathfrak{A} is said to be (Arens) regular. It can be verified that \mathfrak{A} is Arens regular if and only if $WAP(\mathfrak{A}) = \mathfrak{A}^*$. Further information for the Arens regularity of Banach algebras can be found in [5, 6].

WAP-algebras, as a generalization of the Arens regular algebras, has been introduced and intensively studied in [9]. A Banach algebra \mathfrak{A} for which the natural embedding $x \mapsto \hat{x}$ of \mathfrak{A} into $WAP(\mathfrak{A})^*$ where $\hat{x}(\gamma) = \gamma(x)$ for $\gamma \in WAP(\mathfrak{A})$, is bounded below; that is, for

some $m \in \mathbb{R}$ with $m > 0$ we have $\|\hat{x}\| \geq m\|x\|$, is called a WAP-algebra. When \mathfrak{A} is Arens regular or dual Banach algebra, the natural embedding of \mathfrak{A} into $WAP(\mathfrak{A})^*$ is isometric [16, Corollary 4.6]. Also Theorem 3.1 shows that $M_b(S, \omega)$ is a WAP-algebra if and only if this embedding is isometric and of course bounded below, however in general $M_b(S, \omega)$ is neither Arens regular nor dual Banach algebra. It has also known that \mathfrak{A} is a WAP-algebra if and only if it admits an isometric representation on a reflexive Banach space.

Moreover, group algebras are also always WAP-algebras, however; they are neither dual Banach algebras, nor Arens regular in the case where the underlying group is not discrete, see [17]. Ample information about WAP-algebras with further details can be found in the impressive paper [9].

A character on an abelian algebra \mathfrak{A} is a non-zero homomorphism $\tau : \mathfrak{A} \rightarrow \mathbb{C}$. The set of all characters on \mathfrak{A} endowed with relative weak*- topology is called character space of \mathfrak{A} .

Following [3], a semi-topological semigroup is a semigroup S equipped with a Hausdorff topology under which the multiplication of S is separately continuous. If the multiplication of S is jointly continuous, then S is said to be a topological semigroup. We write $\ell^\infty(S)$ for the commutative C^* -algebra of all bounded complex-valued functions on S . In the case where S is locally compact we also write $C(S)$ and $C_0(S)$ for the C^* -subalgebras of $\ell^\infty(S)$ consist of continuous elements and continuous elements which vanish at infinity, respectively. We also denote the space of all *weakly almost periodic* functions on S by $wap(S)$ which is defined by

$$wap(S) = \{f \in C(S) : \{R_s f : s \in S\} \text{ is relatively weakly compact}\},$$

where $R_s f(t) = f(ts)$, $(s, t \in S)$. Then $wap(S)$ is a C^* -subalgebra of $C(S)$ and its character space S^{wap} , endowed with the Gelfand topology, enjoys a (Arens type) multiplication that turns it into a compact semi-topological semigroup. The evaluation mapping $\epsilon : S \rightarrow S^{wap}$ is a homomorphism with dense image and it induces an isometric *-isomorphism from $C(S^{wap})$ onto $wap(S)$. Many other properties of $wap(S)$ and its inclusion relations among other function algebras are completely explored in [3].

Let $M_b(S)$ be the Banach space of all complex regular Borel measures on S , which is known as a Banach algebra with the total variation norm and under the convolution product $*$ defined by the equation

$$\langle \mu * \nu, g \rangle = \int_S \int_S g(xy) d\mu(x) d\nu(y) \quad (g \in C_0(S))$$

and as dual of $C_0(S)$.

Throughout, a weight on S is a Borel measurable function $\omega : S \rightarrow (0, \infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t), \quad (s, t \in S).$$

For $\mu \in M_b(S)$ we define $(\mu\omega)(E) = \int_E \omega d\mu$, ($E \subseteq S$ is Borel set). If $\omega \geq 1$, then

$$M_b(S, \omega) = \{\mu \in M_b(S) : \mu\omega \in M_b(S)\}$$

is known as a Banach algebra which is called the *weighted semigroup measure algebra* (see [6, 12, 13, 14] for further details about such algebras and arbitrary weight functions). Let S be a locally compact semigroup, and let $B(S)$ denote the space of all Borel measurable and bounded functions on S . Set $B(S, 1/\omega) = \{f : S \rightarrow \mathbb{C} : f/\omega \in B(S)\}$. A standard predual for $M_b(S, \omega)$ is

$$C_0(S, 1/\omega) = \{f \in B(S, 1/\omega) : f/\omega \in C_0(S)\}.$$

Let $f \in C(S, 1/\omega)$ then f is called ω -weakly almost periodic if the set $\{\frac{R_s f}{\omega(s)\omega} : s \in S\}$ is relatively weakly compact in $C(S)$, where R_s is defined as above. The set of all ω -weakly almost periodic functions on S is denoted by $wap(S, 1/\omega)$.

In the case where S is discrete we write $\ell_1(S, \omega)$ instead of $M_b(S, \omega)$ and $c_0(S, \frac{1}{\omega})$ instead of $C_0(S, \frac{1}{\omega})$. Then the space

$$\ell_1(S, \omega) = \{f : f = \sum_{s \in S} f(s)\delta_s, \quad \|f\|_{1, \omega} = \sum_{s \in S} |f(s)|\omega(s) < \infty\}$$

(where, $\delta_s \in \ell_1(S, \omega)$ be the point mass at s which can be thought as the vector basis element of $\ell_1(S, \omega)$) equipped with the multiplication

$$f * g = \sum_{r \in S} \sum_{st=r} f(s)g(t)\delta_r$$

(and also define $f * g = 0$ if for each $r \in S$ the equation $st = r$ has no solution;) is a Banach algebra which will be called weighted semigroup algebra. We also suppress 1 from the notation whenever $w = 1$.

2. Semigroup Measure Algebras as Dual Banach Algebras

It is known that the semigroup algebra $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if S is weakly cancellative semigroup, see [7, Theorem 4.6]. Throughout this section ω is a continuous weight on S . This result has been extended for the weighted

semigroup algebras $\ell_1(S, \omega)$; [1, 8]. In this section we extend this results to the non-discrete case. We provide some necessary and sufficient conditions that the measure algebra $M_b(S, \omega)$ becomes a dual Banach algebra with respect to the predual $C_0(S, 1/\omega)$.

Let F and K be nonempty subsets of a semigroup S and $s \in S$. We put

$$s^{-1}F = \{t \in S : st \in F\}, \text{ and } Fs^{-1} = \{t \in S : ts \in F\}$$

and we also write $s^{-1}t$ for the set $s^{-1}\{t\}$, FK^{-1} for $\cup\{Fs^{-1} : s \in K\}$ and $K^{-1}F$ for $\cup\{s^{-1}F : s \in K\}$.

A semigroup S is called left (respectively, right) zero semigroup if $xy = x$ (respectively, $xy = y$), for all $x, y \in S$. A semigroup S is called zero semigroup if there exist $z \in S$ such that $xy = z$ for all $x, y \in S$. A semigroup S is said to be *left (respectively, right) weakly cancellative semigroup* if $s^{-1}F$ (respectively, Fs^{-1}) is finite for each $s \in S$ and each finite subset F of S . A semigroup S is said to be *weakly cancellative semigroup* if it is both left and right weakly cancellative semigroup.

A semi-topological semigroup S is said to be *compactly cancellative semigroup* if for every compact subsets F and K of S the sets $F^{-1}K$ and KF^{-1} are compact set.

Lemma 2.1. Let S be a topological semigroup. For every compact subsets F and K of S the sets $F^{-1}K$ and KF^{-1} are closed.

Proof. If $F^{-1}K$ is empty, then it is closed. Let x be in the closure of $F^{-1}K$. Then there is a net (x_α) in $F^{-1}K$ such that $x_\alpha \rightarrow x$. Since $x_\alpha \in F^{-1}K$ there is a net (f_α) in F such that $f_\alpha x_\alpha \in K$. Using the compactness of F and K , by passing to a subnet, if necessary, we may suppose that $f_\alpha x_\alpha \rightarrow k$ and $f_\alpha \rightarrow f$, for some $f \in F$ and $k \in K$. So $fx = k \in K$, that is $x \in F^{-1}K$. Therefore $F^{-1}K$ is closed. A similar argument shows that KF^{-1} is also closed. \square

In the next result we study $M_b(S, \omega)$ from the dual Banach algebra point of view.

Theorem 2.1. Let S be a locally compact topological semigroup and ω be a continuous weight on S . Then the measure algebra $M_b(S, \omega)$ is a dual Banach algebra with respect to the predual $C_0(S, 1/\omega)$ if and only if for all compact subsets F and K of S , the maps $\frac{\chi_{F^{-1}K}}{\omega}$ and $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity.

Proof. Suppose that $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$ and let K, F be nonempty compact subsets of S with a net (x_α) in $F^{-1}K$. Let $C_{00}^+(S)$ denote the non-negative continuous functions with compact support on S and set $C_{00}^+(S, \frac{1}{\omega}) = \{f \in C_0(S, \frac{1}{\omega}) : f/\omega \in C_{00}^+(S)\}$. Since ω is continuous we may choose $f \in C_{00}^+(S, \frac{1}{\omega})$

with $f(K) = 1$. There is a net $(t_\alpha) \in F$ such that $t_\alpha x_\alpha \in K$ and the compactness of F guaranties the existence of a subnet (t_γ) of (t_α) such that $t_\gamma \rightarrow t_0$ for some t_0 in S . Indeed, for $s \in S$,

$$\lim_{\gamma} \left(\frac{\delta_{t_\gamma} \cdot f}{\omega} \right)(s) = \lim_{\gamma} \frac{f(t_\gamma s)}{\omega(s)} = \frac{f(t_0 s)}{\omega(s)} = \frac{\delta_{t_0} \cdot f}{\omega}(s)$$

there is a γ_0 such that

$$\bigcup_{\gamma \geq \gamma_0} t_\gamma^{-1} K \subseteq \bigcup_{\gamma \geq \gamma_0} \{s \in S : (\frac{\delta_{t_\gamma} \cdot f}{\omega})(s) \geq 1\} \subseteq \{s \in S : (\frac{\delta_{t_0} \cdot f}{\omega})(s) \geq \frac{1}{2}\}.$$

Let $H = \{t_\gamma : \gamma \geq \gamma_0\} \cup \{t_0\}$. Then

$$H^{-1}K = \bigcup_{\gamma \geq \gamma_0} t_\gamma^{-1} K \cup t_0^{-1} K \subseteq \{s \in S : (\frac{\delta_{t_0} \cdot f}{\omega})(s) \geq \frac{1}{2}\}$$

and so $H^{-1}K$ is compact. Furthermore, $t_\gamma x_\gamma \in K$, that is (x_γ) is a net in compact set $H^{-1}K$. This means that (x_α) has a convergent subnet and this is the proof of necessity.

The sufficiency can be adopted from [1, Proposition 3.1] with some modifications. Let $f \in C_0(S, 1/\omega)$, $\mu \in M_b(S, \omega)$ and $\varepsilon > 0$ be arbitrary. There exist compact subsets F and K of S such that $|\frac{f}{\omega}(s)| < \varepsilon$ for all $s \notin K$ and $|(\mu\omega)|(S \setminus F) < \varepsilon$.

Let $s \notin \{t \in F^{-1}K : \omega(t) \leq \frac{1}{\varepsilon}\}$, which is compact by hypothesis. Then

$$\begin{aligned} \left| \frac{\mu \cdot f}{\omega}(s) \right| &= \left| \int_S \frac{f(ts)}{\omega(s)} d\mu(t) \right| \\ &\leq \left| \int_F \frac{f(ts)}{\omega(s)} d\mu(t) \right| + \left| \int_{S \setminus F} \frac{f(ts)}{\omega(s)} d\mu(t) \right| \\ &\leq \int_F \left| \frac{f(ts)}{\omega(ts)} \right| \omega(t) d|\mu|(t) + \int_{S \setminus F} \left| \frac{f(ts)}{\omega(ts)} \right| \omega(t) d|\mu|(t) \\ &\leq \varepsilon \int_S \omega(t) d|\mu|(t) + \|f\|_{\omega, \infty} \int_{S \setminus F} \omega(t) d|\mu|(t) \\ &\leq \varepsilon \|\mu\|_{\omega} + \varepsilon \|f\|_{\omega, \infty} \end{aligned}$$

That is, $\mu \cdot f \in C_0(S, 1/\omega)$. Therefore $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$. \square

The next Corollaries are immediate consequences of Theorem 2.1.

Corollary 2.1. Let S be a locally compact topological semigroup. Then the measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if S is a compactly cancellative semigroup.

Corollary 2.2. [1, Theorem 2.2] For a semigroup S the semigroup algebra $\ell_1(S, \omega)$ is a dual Banach algebra with respect to the predual $c_0(S, 1/\omega)$ if and only if for all $s, t \in S$, the maps $\frac{\chi_{t^{-1}s}}{\omega}$ and $\frac{\chi_{st^{-1}}}{\omega}$ are in $c_0(S)$.

Corollary 2.3. For a locally compact topological semigroup S , if $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ then $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$.

Corollary 2.4. Let S be either a left zero (right zero) or a zero locally compact semigroup. There is a weight ω such that $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, \frac{1}{\omega})$ if and only if S is σ -compact.

Proof. Let K and F be compact subsets of S . It can be readily verified that in either cases (being left zero, right zero or zero) the sets $F^{-1}K$ and KF^{-1} are equal to either empty or S . Put

$$S_m = \{t \in F^{-1}K : \omega(t) \leq m\} = \{t \in S : \omega(t) \leq m\} \quad (m \in \mathbb{N}).$$

Then $S = \cup_{m \in \mathbb{N}} S_m$ and so S is σ -compact. For the converse let $S = \cup_{n \in \mathbb{N}} S_n$ as a disjoint union of compact sets and let z be a (left or right) zero for S . Define $\omega(z) = 1$ and $\omega(x) = 1 + n$ for $x \in S_n$ then ω is a weight on S and $M_b(S, \omega)$ is a dual Banach algebra. \square

Examples 2.1. (1) The set $S = \mathbb{R}^+ \times \mathbb{R}$ equipped with the multiplication

$$(x, y) \cdot (x', y') = (x + x', y') \quad ((x, y), (x', y') \in S)$$

and the weight $\omega(x, y) = e^{-x}(1 + |y|)$ is a weighted semigroup. In this example $[a, b]$ denotes a closed interval. As for $F = [a, b] \times [c, d]$ and $K = [e, f] \times [g, h]$, with $[c, d] \cap [g, h] \neq \emptyset$

$$\begin{aligned} F^{-1}K &= \bigcup_{(x, y) \in F} (x, y)^{-1}K \\ &= \bigcup_{(x, y) \in F} \{(s, t) \in S : (x, y)(s, t) \in K\} \\ &= \bigcup_{(x, y) \in F} \{(s, t) \in S : (x + s, t) \in K\} \\ &= \bigcup_{(x, y) \in F} [e - x, f - x] \times [g, h] = [e - b, f - a] \times [g, h] \end{aligned}$$

and

$$\begin{aligned}
KF^{-1} &= \bigcup_{(x,y) \in F} K(x,y)^{-1} \\
&= \bigcup_{(x,y) \in F} \{(s,t) \in S : (s,t)(x,y) \in K\} \\
&= \bigcup_{(x,y) \in F} \{(s,t) \in S : (x+s,y) \in K\} \\
&= \bigcup_{(x,y) \in F} [e-x, f-x] \times \mathbb{R} = [e-b, f-a] \times \mathbb{R}
\end{aligned}$$

Thus

$$F^{-1}K = [e-b, f-a] \times [g, h] \quad \text{and} \quad KF^{-1} = \begin{cases} [e-b, f-a] \times \mathbb{R} & \text{if } [c, d] \cap [g, h] \neq \emptyset \\ \emptyset & \text{if } [c, d] \cap [g, h] = \emptyset \end{cases},$$

$M_b(S)$ is not a dual Banach algebra by Corollary 2.1. However, for all compact subsets F and K of S , the maps $\frac{\chi_{F^{-1}K}}{\omega}$ and $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity. So $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$. This shows that the converse of Corollary 2.3 may not be valid.

- (2) For the semigroup $S = [0, \infty)$ endowed with the zero multiplication, neither $M_b(S)$ nor $\ell_1(S)$ is a dual Banach algebra. In fact, S is neither compactly nor weakly cancellative semigroup.

3. Semigroup Measure Algebras as WAP-Algebras

In this section, for a weighted locally compact semi-topological semigroup (S, ω) , we investigate some necessary and sufficient condition for $M_b(S, \omega)$ being WAP-algebra. First, we provide some preliminaries.

Definition 3.1. Let $\tilde{\mathcal{F}}$ be a linear subspace of $B(S, 1/\omega)$, and let $\tilde{\mathcal{F}}_r$ denote the set of all real-valued members of $\tilde{\mathcal{F}}$. A mean on $\tilde{\mathcal{F}}$ is a linear functional $\tilde{\mu}$ on $\tilde{\mathcal{F}}$ with the property that

$$\inf_{s \in S} \frac{f}{\omega}(s) \leq \tilde{\mu}(f) \leq \sup_{s \in S} \frac{f}{\omega}(s) \quad (f \in \tilde{\mathcal{F}}_r).$$

The set of all means on $\tilde{\mathcal{F}}$ is denoted by $M(\tilde{\mathcal{F}})$. If $\tilde{\mathcal{F}}$ is also an algebra with the multiplication given by $f \odot g := (f.g)/\omega$ ($f, g \in \tilde{\mathcal{F}}$) and if $\tilde{\mu} \in M(\tilde{\mathcal{F}})$ satisfies

$$\tilde{\mu}(f \odot g) = \tilde{\mu}(f)\tilde{\mu}(g) \quad (f, g \in \tilde{\mathcal{F}}),$$

then $\tilde{\mu}$ is said to be multiplicative. The set of all multiplicative means on $\tilde{\mathcal{F}}$ will be denoted by $MM(\tilde{\mathcal{F}})$.

Let $\tilde{\mathcal{F}}$ be a conjugate closed, linear subspace of $B(S, 1/\omega)$ such that $\omega \in \tilde{\mathcal{F}}$.

- (i) For each $s \in S$ define $\epsilon(s) \in M(\tilde{\mathcal{F}})$ by $\epsilon(s)(f) = (f/\omega)(s)$ ($f \in \tilde{\mathcal{F}}$). The mapping $\epsilon : S \longrightarrow M(\tilde{\mathcal{F}})$ is called the evaluation mapping. If $\tilde{\mathcal{F}}$ is also an algebra, then $\epsilon(S) \subseteq MM(\tilde{\mathcal{F}})$.
- (ii) Let $\tilde{X} = M(\tilde{\mathcal{F}})$ (resp. $\tilde{X} = MM(\tilde{\mathcal{F}})$, if $\tilde{\mathcal{F}}$ is a subalgebra) be endowed with the relative weak* topology. For each $f \in \tilde{\mathcal{F}}$ the function $\hat{f} \in C(\tilde{X})$ is defined by

$$\hat{f}(\tilde{\mu}) := \tilde{\mu}(f) \quad (\tilde{\mu} \in \tilde{X}).$$

Furthermore, we define $\hat{\tilde{\mathcal{F}}} := \{\hat{f} : f \in \tilde{\mathcal{F}}\}$

Remark 3.1. (i) *The mapping $f \longrightarrow \hat{f} : \tilde{\mathcal{F}} \longrightarrow C(\tilde{X})$ is clearly linear and multiplicative if $\tilde{\mathcal{F}}$ is an algebra and $\tilde{X} = MM(\tilde{\mathcal{F}})$. Also it preserves complex conjugation, and is an isometry, since for any $f \in \tilde{\mathcal{F}}$*

$$\begin{aligned} \|\hat{f}\| &= \sup\{|\hat{f}(\tilde{\mu})| : \tilde{\mu} \in \tilde{X}\} = \sup\{|\tilde{\mu}(f)| : \tilde{\mu} \in \tilde{X}\} \\ &= \sup\{|\mu(\frac{f}{\omega})| : \mu \in X\} \leq \sup\{|\mu(\frac{f}{\omega})| : \mu \in C(X)^*, \|\mu\| \leq 1\} \\ &= \|\frac{f}{\omega}\| = \|f\|_{\omega} = \sup\{|\frac{f}{\omega}(s)| : s \in S\} = \sup\{|\epsilon(s)(f)| : s \in S\} \\ &= \sup\{|\hat{f}(\epsilon(s))| : s \in S\} \leq \|\hat{f}\|, \end{aligned}$$

where $X = M(\mathcal{F})$ and $\mathcal{F} = \{f/\omega : f \in \tilde{\mathcal{F}}\}$. Note that $\hat{f}(\epsilon(s)) = \epsilon(s)(f) = (\frac{f}{\omega})(s)$ ($f \in \tilde{\mathcal{F}}, s \in S$). This identity may be written in terms of dual map $\tilde{\epsilon}^* : C(\tilde{X}) \longrightarrow C(S, \omega)$ as $\tilde{\epsilon}^*(\hat{f}) = f$ for $f \in \tilde{\mathcal{F}}$.

- (ii) *Let $\tilde{\mathcal{F}}$ be a conjugate closed linear subspace of $B(S, 1/\omega)$, containing ω . Then $M(\tilde{\mathcal{F}})$ is convex and weak* compact, $co(\epsilon(S))$ is weak* dense in $M(\tilde{\mathcal{F}})$, $\tilde{\mathcal{F}}^*$ is the weak* closed linear span of $\epsilon(S)$, $\epsilon : S \longrightarrow M(\tilde{\mathcal{F}})$ is weak* continuous, and if $\tilde{\mathcal{F}}$ is also an algebra, then $MM(\tilde{\mathcal{F}})$ is weak* compact and $\epsilon(S)$ is weak* dense in $MM(\tilde{\mathcal{F}})$.*
- (iii) *Let $\tilde{\mathcal{F}}$ be a C^* -subalgebra of $B(S, 1/\omega)$, containing ω . If \tilde{X} denotes the space $MM(\tilde{\mathcal{F}})$ with the relative weak* topology, and if $\epsilon : S \longrightarrow \tilde{X}$ denotes the evaluation mapping, then the mapping $f \longrightarrow \hat{f} : \tilde{\mathcal{F}} \longrightarrow C(\tilde{X})$ is an isometric isomorphism with the inverse $\tilde{\epsilon}^* : C(\tilde{X}) \longrightarrow \tilde{\mathcal{F}}$.*

Let $\tilde{\mathcal{F}} = wap(S, 1/\omega)$. Then $\tilde{\mathcal{F}}$ is a C^* -algebra and a subspace of $WAP(M_b(S, \omega))$, see [11, Theorem1.6, Theorem3.3]. Set $\tilde{X} = MM(\tilde{\mathcal{F}})$. By the above remark $wap(S, 1/\omega) \cong$

$C(\tilde{X})$ and so

$$M_b(\tilde{X}) \cong C(\tilde{X})^* \cong \text{wap}(S, 1/\omega)^* \subseteq \text{WAP}(M_b(S, \omega))^*.$$

Let $\epsilon : S \longrightarrow \tilde{X}$ be the evaluation mapping. We also define $\bar{\epsilon} : M_b(S, \omega) \longrightarrow M_b(\tilde{X})$, by $\langle \bar{\epsilon}(\mu), f \rangle = \int_S f \omega d\mu$ for $f \in \text{wap}(S, 1/\omega) \cong C(\tilde{X})$. Then for every Borel set B in \tilde{X} ,

$$\bar{\epsilon}(\mu)(B) = (\mu\omega)(\epsilon^{-1}(B)).$$

In particular, $\bar{\epsilon}(\frac{\delta_x}{\omega(x)}) = \delta_{\epsilon(x)}$.

The next theorem is the main result of this section.

Theorem 3.1. For every weighted locally compact semi-topological semigroup (S, ω) the following statements are equivalent:

- (1) The evaluation map $\epsilon : S \longrightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(\text{wap}(S, 1/\omega))$;
- (2) $\bar{\epsilon} : M_b(S, \omega) \longrightarrow M_b(\tilde{X})$ is an isometric isomorphism;
- (3) $M_b(S, \omega)$ is a WAP-algebra.

Proof. (1) \Rightarrow (2). Take $\mu \in M_b(S, \omega)$, say $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_j \in M_b(S, \omega)^+$. Set $\nu_j = \bar{\epsilon}(\mu_j) \in M_b(\tilde{X})^+$ for $j = 1, 2, 3, 4$, and set

$$\nu = \bar{\epsilon}(\mu) = \nu_1 - \nu_2 + i(\nu_3 - \nu_4).$$

Take $\delta > 0$. For each j , there exists Borel set B_j in \tilde{X} such that $\nu_j(B) \geq 0$ for each Borel subset B of B_j and $\sum_{j=1}^4 \nu_j(B_j) > \|\nu\| - \delta$. In fact, by Hahn decomposition theorem for signed measures $\lambda_1 = \nu_1 - \nu_2$ and $\lambda_2 = \nu_3 - \nu_4$ there exist four Borel sets P_1, P_2, N_1 and N_2 in \tilde{X} such that

$$P_1 \cup N_1 = \tilde{X}, \quad P_1 \cap N_1 = \emptyset, \quad P_2 \cup N_2 = \tilde{X}, \quad P_2 \cap N_2 = \emptyset$$

and for every Borel set E of \tilde{X} we have,

$$\nu_1(E) = \lambda_1(P_1 \cap E), \quad \nu_2(E) = -\lambda_1(N_1 \cap E), \quad \nu_3(E) = \lambda_2(P_2 \cap E), \quad \nu_4(E) = -\lambda_2(N_2 \cap E).$$

that is $\nu_1, \nu_2, \nu_3, \nu_4$ are concentrated respectively on P_1, N_1, P_2, N_2 .

Set $D_1 := P_1 \cap N_2, D_2 := N_1 \cap P_2, D_3 := P_2 \cap P_1, D_4 := N_2 \cap N_1$. Then the family $\{D_1, D_2, D_3, D_4\}$ is a partition of \tilde{X} . Also for $\delta > 0$ there is a compact set K for which

$$\|\nu\| - \delta \leq \sum_{j=1}^4 \|\nu_j|_{D_j}\| - \delta \leq \sum_{j=1}^4 \nu_j|_{D_j}(K) = \sum_{j=1}^4 \nu_j(D_j \cap K).$$

Set $B_j = D_j \cap K$. Then the sets B_1, B_2, B_3, B_4 are pairwise disjoint.

Set $C_j = (\epsilon)^{-1}(B_j)$, a Borel set in S . Then $(\mu_j \omega)(C_j) = \nu_j(B_j)$. Since ϵ is injection, the sets C_1, C_2, C_3, C_4 are pairwise disjoint, and so

$$\|\mu\|_\omega \geq \sum_{j=1}^4 |\mu \omega(C_j)| \geq \sum_{j=1}^4 (\mu_j \omega)(C_j) = \sum_{j=1}^4 \nu_j(B_j) > \|\nu\| - \delta$$

This holds for each $\delta > 0$, so $\|\mu\|_\omega \geq \|\nu\|$. A similar argument shows that $\|\mu\|_\omega \leq \|\nu\|$. Thus $\|\mu\|_\omega = \|\nu\|$.

(2) \Rightarrow (1). Let $P(S, \omega)$ denote the subspace of all probability measures of $M_b(S, \omega)$ and $\text{ext}(P(S, \omega))$ the extreme points of unit ball of $P(S, \omega)$. Then $\text{ext}(P(S, \omega)) = \{\frac{\delta_x}{\omega(x)} : x \in S\} \cong S$ and $\text{ext}(P(\tilde{X})) \cong \tilde{X}$, see [4, p.151]. By injectivity of $\bar{\epsilon}$, it maps the extreme points of the unit ball onto the extreme points of the unit ball, thus $\epsilon : S \longrightarrow \tilde{X}$ is a one to one map.

(2) \Rightarrow (3). Since \tilde{X} is compact, $M_b(\tilde{X})$ is a dual Banach algebra with respect to $C(\tilde{X})$, so it has an isometric representation ψ on a reflexive Banach space E , see [9]. In the following commutative diagram,

$$\begin{array}{ccc} M_b(S, \omega) & \xrightarrow{\bar{\epsilon}} & M_b(\tilde{X}) \\ & \searrow \phi & \downarrow \psi \\ & & B(E) \end{array}$$

If $\bar{\epsilon}$ is isometric, then so is ϕ .

Thus $M_b(S, \omega)$ has an isometric representation on a reflexive Banach space E if $\bar{\epsilon}$ is an isometric isomorphism. So $M_b(S, \omega)$ is a WAP-algebra if $\bar{\epsilon}$ is an isometric isomorphism.

(3) \Rightarrow (1). Let $M_b(S, \omega)$ be a WAP-algebra. Since $\ell_1(S, \omega)$ is a norm closed subalgebra of $M_b(S, \omega)$, then $\ell_1(S, \omega)$ is a WAP-algebra. Using the double limit criterion, it is a simple matter to check that $\text{wap}(S, 1/\omega) = \text{WAP}(\ell_1(S, \omega))$ (see also [11, Theorem 3.7]) where we treat $\ell^\infty(S, 1/\omega)$ as an $\ell_1(S, \omega)$ -bimodule. Then $\bar{\epsilon} : \ell_1(S, \omega) \longrightarrow \text{wap}(S, 1/\omega)^*$ is an isometric isomorphism. Since $\text{wap}(S, 1/\omega)$ is a C^* -algebra, as (2) \Rightarrow (1), $\epsilon : S \longrightarrow \tilde{X}$ is one to one. \square

Corollary 3.1. The following statement are equivalent.

- (1) $\ell_1(S, \omega)$ is a WAP-algebra;
- (2) $M_b(S, \omega)$ is a WAP-algebra.

For $\omega = 1$, it is clear that $\tilde{X} = S^{\text{wap}}$, and the map $\epsilon : S \longrightarrow S^{\text{wap}}$ is one to one if and only if $\text{wap}(S)$ separates the points of S , see [3].

Corollary 3.2. For a locally compact semi-topological semigroup S , the following statements are equivalent:

- (1) $M_b(S)$ is a WAP-algebra;
- (2) $\ell_1(S)$ is a WAP-algebra;
- (3) The evaluation map $\epsilon : S \longrightarrow S^{wap}$ is one to one;
- (4) $wap(S)$ separates the points of S .

Definition 3.2. Let X, Y be sets and f be a complex-valued function on $X \times Y$.

- (1) We say that f is a cluster on $X \times Y$ if for each pair of sequences $(x_n), (y_m)$ of distinct elements of X, Y , respectively

$$\lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m) \quad (1)$$

whenever both sides of (1) exist.

- (2) If f is cluster and both sides of 1 are zero (respectively positive) in all cases, we say that f is 0-cluster (respectively positive cluster).

In general $\{f\omega : f \in wap(S)\} \neq wap(S, 1/\omega)$. By using [2, Lemma1.4] the following is immediate.

Lemma 3.1. Let $\Omega(x, y) = \frac{\omega(xy)}{\omega(x)\omega(y)}$, for $x, y \in S$. Then

- (1) If Ω is cluster, then $\{f\omega : f \in wap(S)\} \subseteq wap(S, 1/\omega)$;
- (2) If Ω is positive cluster, then $wap(S, 1/\omega) = \{f\omega : f \in wap(S)\}$.

It should be noted that if $M_b(S)$ is Arens regular (resp. dual Banach algebra) then $M_b(S, \omega)$ is so. We don't know that if $M_b(S)$ is WAP-algebra, then $M_b(S, \omega)$ is so. The following Lemma give a partial answer to this question.

Corollary 3.3. Let S be a locally compact topological semigroup with a Borel measurable weight function ω such that Ω is cluster on $S \times S$.

- (1) If $M_b(S)$ is a WAP-algebra, then so is $M_b(S, \omega)$;
- (2) If $\ell_1(S)$ is a WAP-algebra, then so is $\ell_1(S, \omega)$.

Proof. (1) Suppose that $M_b(S)$ is a WAP-algebra so $wap(S)$ separates the points of S . By lemma 3.1 for every $f \in wap(S)$, $f\omega \in wap(S, 1/\omega)$. Thus the evaluation map $\epsilon : S \longrightarrow \tilde{X}$ is one to one.

(2) follows from (1). □

Corollary 3.4. For a locally compact semi-topological semigroup S ,

- (1) If $C_0(S) \subseteq \text{wap}(S)$, then the measure algebra $M_b(S)$ is a WAP-algebra.
- (2) If S is discrete and $c_0(S) \subseteq \text{wap}(S)$, then $\ell_1(S)$ is a WAP-algebra.

Proof. (1) By [3, Corollary 4.2.13] the map $\epsilon : S \longrightarrow S^{\text{wap}}$ is one to one, thus $M_b(S)$ is a WAP-algebra.

(2) follows from (1). □

Dales, Lau and Strauss [7, Theorem 4.6, Proposition 8.3] showed that for a semigroup S , $\ell^1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if S is weakly cancellative. If S is left or right weakly cancellative semigroup, then $\ell^1(S)$ is a WAP-algebra. The next example shows that the converse is not true, in general.

Example 3.1. Let $S = (\mathbb{N}, \min)$ then $\text{wap}(S) = c_0(S) \oplus \mathbb{C}$. So $\ell^1(S)$ is a WAP-algebra but S is neither left nor right weakly cancellative. In fact, for $f \in \text{wap}(S)$ and all sequences $\{a_n\}$, $\{b_m\}$ with distinct element in S , we have $\lim_m f(b_m) = \lim_m \lim_n f(a_n b_m) = \lambda = \lim_n \lim_m f(a_n b_m) = \lim_n f(a_n)$, for some $\lambda \in \mathbb{C}$. This means $f - \lambda \in c_0(S)$ and $\text{wap}(S) \subseteq c_0(S) \oplus \mathbb{C}$. The other inclusion is clear.

If $\{x_n\}$ and $\{y_m\}$ are sequences in S we obtain an infinite matrix $\{x_n y_m\}$ which has $x_n y_m$ as its entry in the m th row and n th column. As in [2], a matrix is said to be of row type C (resp. column type C) if the rows (resp. columns) of the matrix are all constant and distinct. A matrix is of type C if it is constant or of row or column type C .

J.W.Baker and A. Rejali in [2, Theorem 2.7(v)] showed that $\ell^1(S)$ is Arens regular if and only if for each pair of sequences $\{x_n\}$, $\{y_m\}$ with distinct elements in S there is a submatrix of $\{x_n y_m\}$ of type C .

A matrix $\{x_n y_m\}$ is said to be upper triangular constant if $x_n y_m = s$ if and only if $m \geq n$ and it is lower triangular constant if $x_n y_m = s$ if and only if $m \leq n$. A matrix $\{x_n y_m\}$ is said to be W -type if every submatrix of $\{x_n y_m\}$ is neither upper triangular constant nor lower triangular constant.

Theorem 3.2. Let S be a semigroup. The following statements are equivalent:

- (1) $c_0(S) \subseteq \text{wap}(S)$.
- (2) For each $s \in S$ and each pair $\{x_n\}$, $\{y_m\}$ of sequences in S ,

$$\{\chi_s(x_n y_m) : n < m\} \cap \{\chi_s(x_n y_m) : n > m\} \neq \emptyset;$$

- (3) For each pair $\{x_n\}$, $\{y_m\}$ of sequences in S with distinct elements, $\{x_n y_m\}$ is a W -type matrix;

- (4) For every $s \in S$, every infinite set $B \subset S$ contains a finite subset F such that $\cap\{sb^{-1} : b \in F\} \setminus (\cap\{sb^{-1} : b \in B \setminus F\})$ and $\cap\{b^{-1}s : b \in F\} \setminus (\cap\{b^{-1}s : b \in B \setminus F\})$ are finite.

Proof. (1) \Leftrightarrow (2). For all $s \in S$, $\chi_s \in wap(S)$ if and only if

$$\{\chi_s(x_n y_m) : n < m\} \cap \{\chi_s(x_n y_m) : n > m\} \neq \emptyset.$$

(3) \Rightarrow (1) Let $c_0(S) \not\subseteq wap(S)$ then there are sequences $\{x_n\}, \{y_m\}$ in S with distinct elements such that for some $s \in S$,

$$1 = \lim_m \lim_n \chi_s(x_n y_m) \neq \lim_n \lim_m \chi_s(x_n y_m) = 0.$$

Since $\lim_n \lim_m \chi_s(x_n y_m) = 0$, for $1 > \varepsilon > 0$ there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $\lim_m \chi_s(x_n y_m) < \varepsilon$. This implies for all $n \geq N$, $\lim_m \chi_s(x_n y_m) = 0$. Then for $n \geq N$, $1 > \varepsilon > 0$ there is a $M_n \in \mathbb{N}$ such that for all $m \geq M_n$ we have $\chi_s(x_n y_m) < \varepsilon$. So if we omit finitely many terms, for all $n \in \mathbb{N}$ there is $M_n \in \mathbb{N}$ such that for all $m \geq M_n$ we have $x_m y_n \neq s$. As a similar argument, for all $m \in \mathbb{N}$ there is $N_m \in \mathbb{N}$ such that for all $n \geq N_m$, $x_m y_n = s$.

Let $a_1 = x_1, b_1$ be the first y_n such that $a_1 y_n = s$. Suppose a_m, b_n have been chosen for $1 \leq m, n < r$, so that $a_n b_m = s$ if and only if $n \geq m$. Pick a_r to be the first x_m not belonging to the finite set $\cup_{1 \leq n \leq r} \{x_m : x_m y_n = s\}$. Then $a_r b_n \neq s$ for $n < r$. Pick b_r to be the first y_n belonging to the cofinite set $\cap_{1 \leq n \leq r} \{y_n : x_m y_n = s\}$. Then $a_n b_m = s$ if and only if $n \geq m$. The sequences $(a_m), (b_n)$ so constructed satisfy $a_m b_n = s$ if and only if $n \geq m$. That is, $\{a_n b_m\}$ is not of W -type and this is a contradiction.

(1) \Rightarrow (3). Let there are sequences $\{x_n\}, \{y_m\}$ in S such that $\{x_n y_m\}$ is not a W -type matrix, (say) $x_n y_m = s$ if and only if $m \leq n$. Then

$$1 = \lim_m \lim_n \chi_s(x_n y_m) \neq \lim_n \lim_m \chi_s(x_n y_m) = 0.$$

So $\chi_s \notin wap(S)$. Thus $c_0(S) \not\subseteq wap(S)$.

(4) \Leftrightarrow (1) This is Ruppert criterion for $\chi_s \in wap(S)$, see [15, Theorem 4]. □

We conclude with some examples which show that some of the above results cannot be improved.

Examples 3.1.

- (i) Let $S = \mathbb{N}$. Then for S equipped with min multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra but is not neither Arens regular nor a dual Banach algebra. While,

if we replace the min multiplication with max then $\ell_1(S)$ is a dual Banach algebra (so a WAP-algebra) which is not Arens regular. If we change the multiplication of S to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

- (ii) Let S be the set of all sequences with 0, 1 values. We equip S with coordinate wise multiplication. We denote by e_n the sequence with all zero unless a 1 in the n -th place. Let $s = \{x_n\} \in S$, and let $F_w(S)$ be the set of all elements of S such that $x_i = 0$ for only finitely index i . It is easy to see that $F_w(S)$ is countable. Let $F_w(S) = \{s_1, s_2, \dots\}$. Recall that, every element $g \in \ell^\infty(S)$ can be denoted by $g = \sum_{s \in S} g(s)\chi_s$, see [6, p.65]. Suppose

$$g = \sum_{s \in S \setminus F_w(S)} g(s)\chi_s$$

be in $wap(S)$, we show that $g = 0$. Let $s = \{x_n\} \in S$, and $\{k \in \mathbb{N} : x_k = 0\} = \{k_1, k_2, \dots\}$ be an infinite set. Put $a_n = s + \sum_{j=1}^n e_{k_j}$ and $b_m = s + \sum_{i=m}^\infty e_{k_i}$. Then

$$a_n b_m = \begin{cases} \sum_{j=m}^n e_{k_j} + s & \text{if } m \leq n \\ s & \text{if } m > n \end{cases}$$

Thus $g(s) = \lim_n \lim_m g(a_n b_m) = \lim_m \lim_n g(a_n b_m) = \lim_m g(s + \sum_{i=m}^\infty e_{k_i}) = 0$.

In fact,

$$wap(S) = \{f \in \ell^\infty(S) : f = \sum_{i=1}^\infty f(s_i)\chi_{s_i}, \quad s_i \in F_w(S)\} \oplus \mathbb{C}$$

It is clear that $F_w(S)$ is the subsemigroup of S and $wap(F_w(S)) = \ell^\infty(F_w(S))$. So $\ell^1(F_w(S))$ is Arens regular. Let T consists of those sequences $s = \{x_n\} \in S$ such that $x_i = 0$ for infinitely index i , then T is a subsemigroup of S and $wap(T) = \mathbb{C}$. Since $\epsilon|_T : T \rightarrow S^{wap}$ isn't one to one, $\ell^1(S)$ is not a WAP-algebra. This shows that in general $\ell^1(S)$ need not be a WAP-algebra.

- (iii) If we equip $S = \mathbb{R}^2$ with the multiplication $(x, y) \cdot (x', y') = (xx', x'y + y')$, then $M_b(S)$ is not a WAP-algebra. Indeed, every non-constant function f over x -axis is not in $wap(S)$. Let $f(0, z_1) \neq f(0, z_2)$ and $\{x_m\}, \{y_m\}, \{\beta_n\}$ be sequences with distinct elements satisfying the recursive equation

$$\beta_n x_m + y_m = \frac{m z_1 + n z_2}{m + n}$$

Then

$$\begin{aligned} \lim_n \lim_m f((0, \beta_n) \cdot (x_m, y_m)) &= \lim_n \lim_m f(0, \beta_n x_m + y_m) \\ &= \lim_n \lim_m f(0, \frac{m z_1 + n z_2}{m + n}) \\ &= f(0, z_1) \end{aligned}$$

and similarly

$$\lim_m \lim_n f((0, \beta_n) \cdot (x_m, y_m)) = f(0, z_2).$$

Thus the map $\epsilon : S \longrightarrow S^{wap}$ isn't one to one, so $M_b(S)$ is not a WAP-algebra. This shows that in general $M_b(S)$ need not be a WAP-algebra.

- (iv) Let S be the interval $[\frac{1}{2}, 1]$ with multiplication $x.y = \max\{\frac{1}{2}, xy\}$, where xy is the ordinary multiplication on \mathbb{R} . Then for all $s \in S \setminus \{\frac{1}{2}\}$, $x \in S$, $x^{-1}s$ is finite. But $x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}]$. Let $B = [\frac{1}{2}, \frac{3}{4}]$. Then for all finite subset F of B ,

$$\bigcap_{x \in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x \in B \setminus F} x^{-1}\frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}]$$

where $x_F = \max F$. By [15, Theorem 4] $\chi_{\frac{1}{2}} \notin \text{wap}(S)$. So $c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \subsetneq \text{wap}(S)$. It can be readily verified that $\epsilon : S \longrightarrow S^{wap}$ is one to one, so $\ell_1(S)$ is a WAP-algebra but $c_0(S) \not\subseteq \text{wap}(S)$. This is a counter example for the converse of Corollary 3.4.

- (v) Take $T = (\mathbb{N} \cup \{0\}, \cdot)$ with 0 as zero of T and the multiplication defined by

$$n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = T \times T$ is a semigroup with coordinate wise multiplication. Now let $X = \{(k, 0) : k \in T\}$, $Y = \{(0, k) : k \in T\}$ and $Z = X \cup Y$. We use the Ruppert criterion [15] to show that $\chi_z \notin \text{wap}(S)$, for each $z \in Z$. Let $B = \{(k, n) : k, n \in T\}$, then $(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}$. Thus for all finite subsets F of B ,

$$\begin{aligned} (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in F\}) &\setminus (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\ &= (\cap \{(k, 0)(k, n)^{-1} : (k, n) \in F\}) \\ &\setminus (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\ &= (B \setminus F) \setminus F = B \setminus F \end{aligned}$$

and the last set is infinite. This means $\chi_{(k, 0)} \notin \text{wap}(S)$. Similarly $\chi_{(0, k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0} f(0, n)\chi_{(0, n)} + \sum_{m=1}^{\infty} f(m, 0)\chi_{(m, 0)}$ be in $\text{wap}(S)$. For arbitrary

fixed n and sequence $\{(n, k)\}$ in S , we have $\lim_k f(n, k) = \lim_k \lim_l f(n, l, k) = \lim_l \lim_k f(n, l, k) = f(n, 0)$ implies $f(n, 0) = 0$. Similarly $f(0, n) = 0$ and $f(0, 0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Since $\text{wap}(S)$ can not separate the points of S so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n, m) = 2^n 3^m$ for $(n, m) \in S$. Then ω is a weight on S such that $\omega \in \text{wap}(S, 1/\omega)$, so the evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one. This means $\ell_1(S, \omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counter example for the converse of Corollary 3.3.

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